Fontene theorems and some corollaries

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Abstract

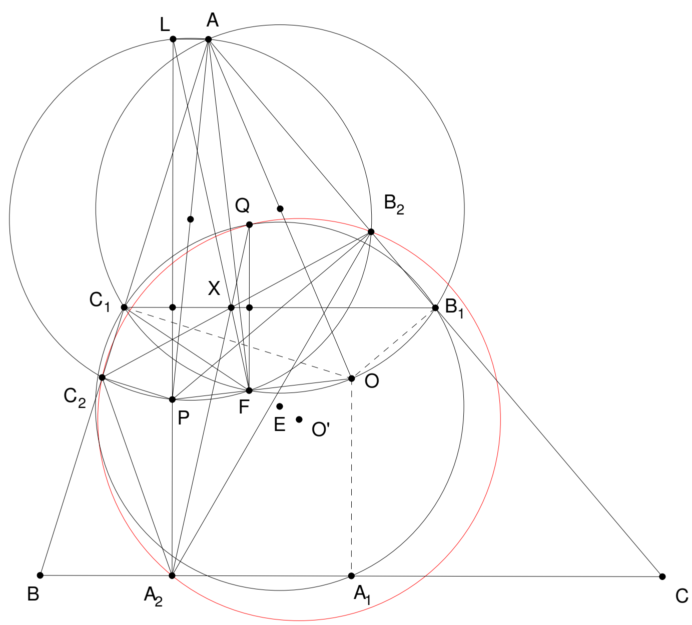
In 1905 and 1906, Fontene. G presented three theorems about the generalization of Feuerbach point in the magazine "Nouvelles Annales de Mathmatiques ". In this note, we give their proofs and learn about their corollaries.

1 Theorems

Fontene theorem 1.

Given triangle ABC. Let P be an arbitrary point in the plane. A_1, B_1, C_1 are the midpoints of BC, CA, AB; $A_2B_2C_2$ is the pedal triangle of P with respect to triangle ABC. Let X, Y, Z be the intersections of B_1C_1 and B_2C_2, A_1C_1 and A_2C_2, A_1B_1 and A_2B_2. Then A_2X, B_2Y, C_2Z concur at the intersection of $(A_1B_1C_1)$ and $(A_2B_2C_2)$.

Proof by Bricard (ibid., 1906).



Let E be the center of $(A_1B_1C_1)$, O' be the center of $(A_2B_2C_2)$, F be the intersection of OP and the circle with diameter OA, L be the reflection of A_2 with respect to B_1C_1 then $AL//BC$. We obtain $\angle ALP = 90^o$.

Since $\angle AFP = \angle AB_2P = \angle AC_2P = \angle ALP = 90^o$ we get L, F, B_2, C_2 are on (AP).

$\angle FC_1X = \angle FAB_1 = \angle B_2C_2F$ then FXC_1C_2 is a cyclic quadrilateral.

Denote L' the intersection of FX and (AP). We have $AL'C_2F$ is a cyclic quadrilateral. But FXC_1C_2 is also cyclic therefore $AL'//B_1C_1$ or $L' \equiv L$, which follows that L, X, F are collinear.

Denote Q the intersection of A_2X and (E). F' is the reflection of Q with respect to B_1C_1.

Consider the Symmetry $\mathcal{S}_{B_1C_1} : (AO) \mapsto (E)$, but $Q \in (E)$ hence $F' \in (AO)$.

On the other side, $\mathcal{S}_{B_1C_1}$ maps A_2 to L. Furthermore A_2, X, Q are collinear so L, X, F' are collinear, which is equivalent to $F' \equiv F$.

We deduce that A_2LQF is a isosceles trapezoid.

This means $\overline{XQ}.\overline{XA_2} = \overline{XL}.\overline{XF} = \overline{XB_2}.\overline{XC_2}$.

Therefore Q lies on (O'). Similarly B_2Y, C_2Z also pass through Q. We are done.

Fontene theorem 2.

If a point P moves on the fixed line d which passes through the circumcenter O of triangle ABC then the pedal circle of P with respect to triangle ABC intersects the Nine-point circle of triangle ABC at a fixed point.

Proof.

According to the proof of Fontene theorem 1, the point of contact Q of (E) and (O') is the reflection of a point F which lies on OP with respect to the line B_1C_1. It is easy to show that O is the orthocenter of triangle $A_1B_1C_1$ thus Q is the Anti-Steiner point of d. Therefore Q is fixed. Our proof is completed.

Fontene theorem 3.

Denote the isogonal conjugate of P with respect to triangle ABC as P'. Then the pedal circle of P is tangent to the Nine-point circle of triangle ABC if and only if O, P, P' are collinear.

Proof.

According to Fontene theorem 2 we can prove that the second intersection Q' of (O') and (E) is the Anti-Steiner point of OP'. This means $Q' \equiv Q$ if and only if $OP \equiv OP'$ or O, P, P' are collinear. We are done.

Note. Feuerbach point is a corollary of Fontene theorem 3, when P coincides with the incenter or 3 excenters.

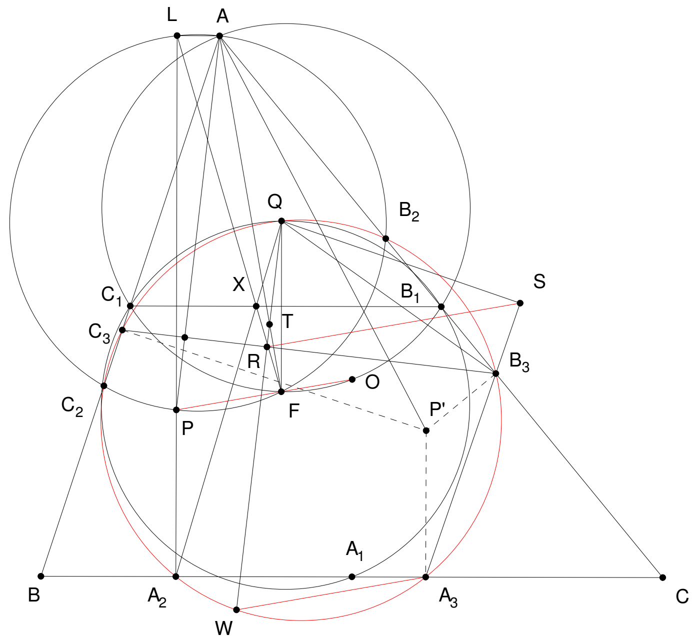
2 Some corollaries

Corollary 1. O' is the orthocenter of triangle XYZ.

Proof. In fact, applying Fontene theorem 1 we claim A_2X, B_2Y, C_2Z concur at a point Q which lies on (O'), which implies that XZ is the polar of Y with respect to $(O'), XY$ is the polar of Z with respect to (O'). Therefore O' is the orthocenter of triangle XYZ.

Corollary 2. Denote $A_3B_3C_3$ the pedal triangle of the isogonal conjugate P' of P with respect to triangle ABC. Then the Simson line of Q with respect to triangle $A_3B_3C_3$ is parallel to the Simson line of Q with respect to triangle $A_1B_1C_1$.

Proof.



Let R, S be the projections of Q on B_3C_3, A_3B_3. Extend QR to W which lies on (O'), $AF \cap QW = \{T\}$. Since $\angle QRS = \angle QB_3S = \angle QWA_3$ we deduce that $RS//A_3W$.

Moreover, $\angle P'C_3B_3 = \angle P'AB_3 = \angle PAB$ hence $AP \perp B_3C_3$, which implies that $AP//QW$.

This means $\angle TWA_3 + \angle FTW = \angle QA_2A_3 + \angle PAF = \angle QA_2A_3 + \angle PLF = \angle QA_2A_3 + \angle PA_2Q = 90^o$. So $SR//A_3W//OP$.(1)

According to the proof of Fontene theorem 2, Q is the Anti-Steiner point of OP with respect to triangle $A_1B_1C_1$ so the Simson line of Q with respect to triangle $A_1B_1C_1$ is parallel to OP.(2)

From (1) and (2) we are done.

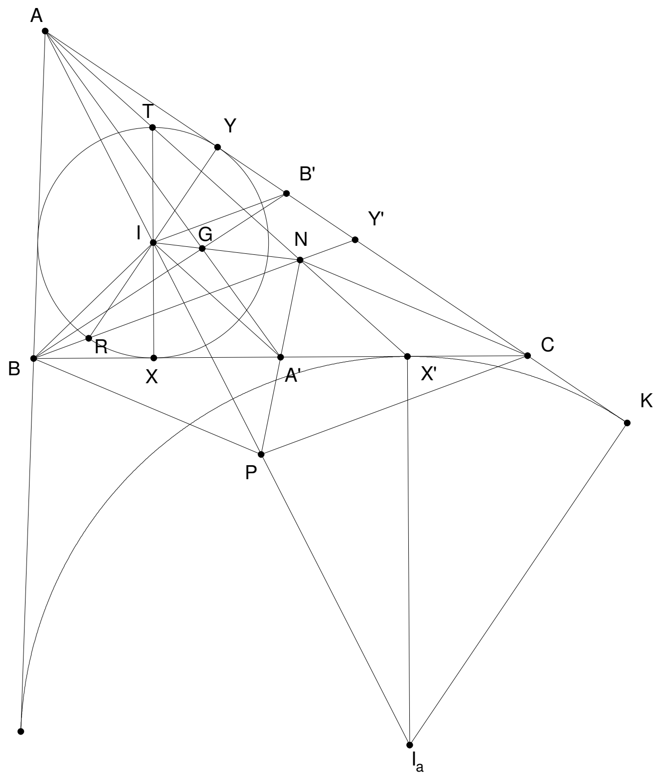
Corollary 3. Given triangle ABC with its circumcircle (O) and its orthocenter H. Let N be the Nagel point of triangle ABC. ON meets (O) at Q. Then the Simson line of Q with respect to triangle ABC is parallel to NH.

Proof.

First let us introduce a lemma:

Lemma 1. Let I, G, N be the incenter, centroid and Nagel point of triangle ABC, respectively. Then I, G, N are collinear and $\overline{IN} = 3\overline{IG}$.

Proof.



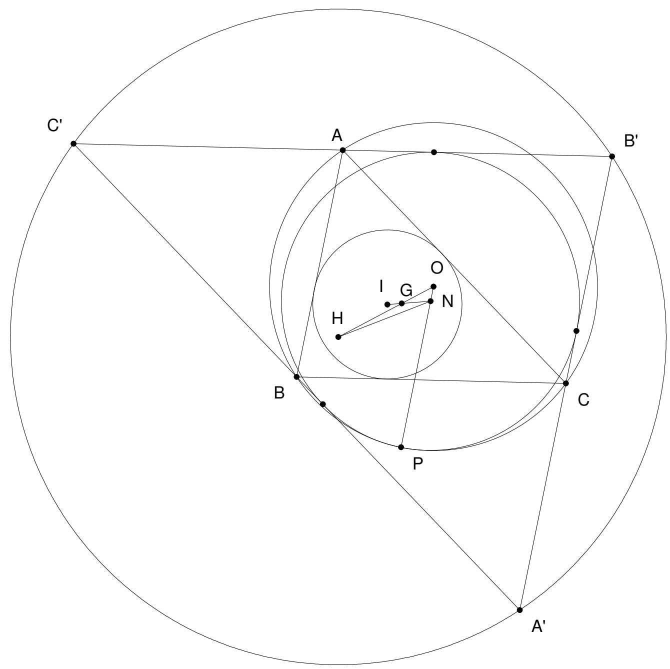
Let X, Y be the tangencies of (I) with BC, AC. XI cuts (I) at T. We will show that A, T, N are collinear.

In fact, let I_a be the A-excenter. (I_a) contacts BC, AC at X', K. We have $\frac{IT}{I_aX'} = \frac{IY}{I_aK} = \frac{IA}{I_aA}$, which follows that A, T, X' are collinear or A, T, N are collinear.

Let P be a point on AI such that I is the midpoint of AP. Let B' be the midpoint of AC. IY intersects (I) at R, BR intersects AC at Y'. Because IB' is the midline of two triangles YRY' and APC at the same time hence $BN//IB'//PC'$. Likewise, $CN//BP$. Therefore $BNCP$ is a parallelogram. We conclude that A' is the midpoint of NP.

Let G' be the intersection of AA' and IN. Since IA' is the midline of triangle APN then $\frac{GA}{GA'} = \frac{GI}{GN} = 2$. Our lemma is solved.

Back to our problem.



Through A, B, C construct three lines which are parallel to opposite side, they intersect each other and make triangle $A'B'C'$. We have (ABC) is the Nine-point circle of triangle $A'B'C'$.

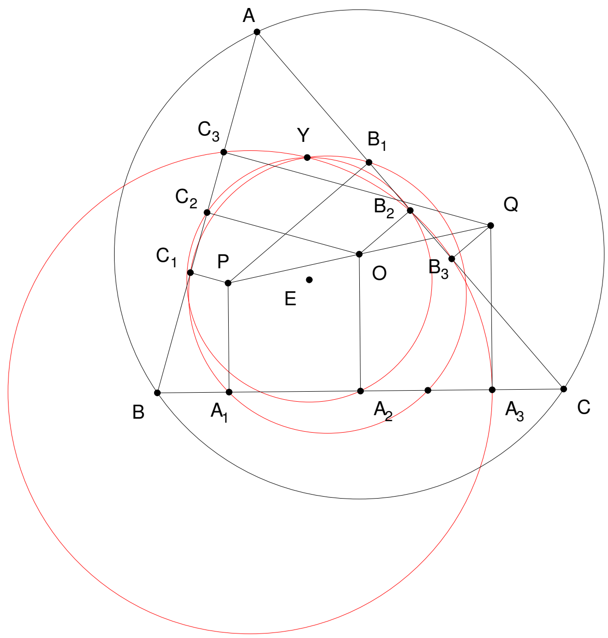
Consider the homothetic $\mathcal{H}^{-2} : (ABC) \mapsto (A'B'C'), O \mapsto H, I \mapsto N$. But O is the center of (ABC) so H is the center of $(A'B'C')$, I is the incenter of triangle ABC so N is the incenter of triangle $A'B'C'$. We get P is the Feuerbach point of triangle $A'B'C'$. According to the proof of Fontene theorem 2 , P is the Anti-Steiner point of NH with respect to triangle ABC. This means the Simson line of P with respect to triangle ABC is parallel to NH.

Corollary 4. Given triangle ABC with its circumcenter O. Let l be a line which passes through O. l intersects BC, CA, AB at X, Y, Z, repspectively. Then four circles $(AX), (BY), (CZ)$, the Nine-point circle of triangle ABC are concurrent.

Proof. In fact, the result follows immediately from Fontene theorem 2 because X, Y, Z are three special cases of P on the line l.

Corollary 5. Given triangle $ABC.P$ is an arbitrary point in the plane. $A_1B_1C_1$ is the pedal triangle of P with respect to ΔABC. A_2, B_2, C_2 are the midpoints of BC, CA, AB, respectively. A_3, B_3, C_3 are the reflections of A_1, B_1, C_1 with respect to A_2, B_2, C_2, respectively. Then three circles $(A_1B_1C_1), (A_2B_2C_2), (A_3B_3C_3)$ are concurrent.

Proof.



Since $A_1B_1C_1$ is the pedal triangle of P with respect to triangle ABC then applying Carnot theorem we obtain:

$BA_1^2 - CA_1^2 + CB_1^2 - AB_1^2 + AC_1^2 - BC_1^2 = 0$.

Hence $CA_3^2 - BA_3^2 + BC_3^2 - AC_3^2 + AB_3^2 - CB_3^2 = 0$.

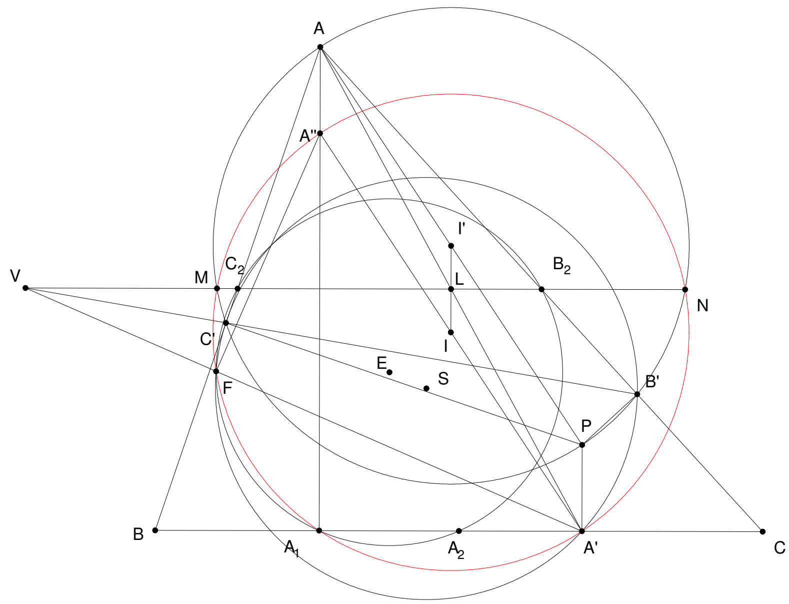
We deduce that $A_3B_3C_3$ is the pedal triangle of Q with respect to triangle ABC.

Applying Thales theorem, it is easy to see that the perpendicular bisector of the line segment BC passes through the midpoint of PQ. Similarly we get the circumcenter O of triangle ABC is the midpoint of PQ.

Denote Y the Anti-Steiner point of OP with respect to triangle $A_2B_2C_2$ then according to the proof of Fontene theorem 2, $Y \in (A_1B_1C_1)$. Moreover O, P, Q are collinear so Y is the Anti-Steiner point of OQ with respect to triangle $A_2B_2C_2$. But $A_3B_3C_3$ is the pedal triangle of Q with respect to triangle ABC thus according to the proof of Fontene theorem 2 again, $Y \in (A_3B_3C_3)$. We are done.

Corollary 6. Given triangle ABC. Let A_1 be the projection of A on BC, A_2, B_2, C_2 be the midpoints of BC, CA, AB, respectively. P is an arbitrary point in the plane, $A'B'C'$ is the pedal triangle of P with respect to $\Delta ABC.(A'B'C') \cap (A_2B_2C_2) = \{F, F'\}$. A line through A' and parallel to AP meets AA_1 at A''. Then $(A'A'')$ passes through one of two points F, F'.

Proof.



Let V be the intersection of $B'C'$ and B_2C_2. According to Fontene theorem 1 we get A', V, F are collinear.

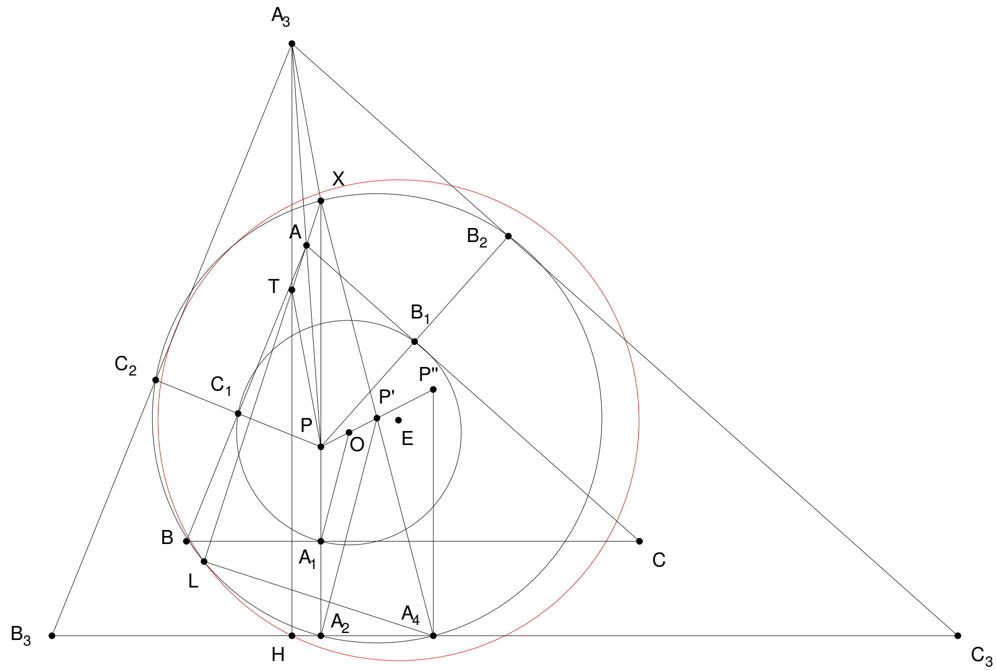
On the other side, denote I', L the midpoints of AP, AA', respectively.

Since $AA''A'P$ is a parallelogram we claim I, L, I' are collinear and $II' \perp BC$. But $L \in B_2C_2$ we deduce that I' is the reflection of I with respect to B_2C_2. Therefore $(I', I'A)$ is the reflections of (I, IA'') with respect to B_2C_2. This means the intersections M, N of two circles lies on B_2C_2.

Moreover $B', C' \in (I', I'A)$ hence $\overline{VF}.\overline{VA'} = \overline{VB'}.\overline{VC'} = \overline{VM}.\overline{VN}$, which follows that F, A', M, N are concyclic or (I, IA'') passes through F.

Corollary 7. Given triangle ABC. Let P be an arbitrary point in the plane, $A_1B_1C_1$ be the pedal triangle of P with respect to ΔABC. Denote the radius of$(A_1B_1C_1)$ as R. P' is the isogonal conjugate of P with respect to ΔABC. Rays A_1P, B_1P, C_1P cut $(P', 2R)$ at X, Y, Z, respectively. Then AX, BY, CZ concur at a point on $(P', 2R)$.

Proof.



Let A_2, B_2, C_2 be the reflections of P with respect to BC, CA, AB. Since the center O of $(A_1B_1C_1)$ is the midpoint of PP' then OA_1 is the midline of triangle $PP'A_2$. This means $P'A_2 = 2R$ or $A_2 \in (P')$. Similarly, $B_2, C_2 \in (P')$.

Consider the homothetic $\mathcal{H}_P^2 : A_1 \mapsto A_2, B_1 \mapsto B_2, C_1 \mapsto C_2, A \mapsto A_3, B \mapsto B_2, C \mapsto C_2$ so $(P', 2R)$ is the image of (O, R) under $\mathcal{H}_P^2$ and (P') is the pedal circle of P with respect to $\Delta A_3B_3C_3$.

Let P'' be the reflection of P with respect to P', A_4 be the projection of P'' on B_3C_3 then $A_4 \in (P')$. Let H be the reflection of A_4 on B_3C_3, T be a point on A_3H such that $A_3T = P''A_3$. According to corollary 6, (A_4T) passes through the intersection L of the Nine-point circle of triangle $A_3B_3C_3$ and (P'), which implies that $TL \perp A_4L$.

On the other side, ray A_1P intersects (P') at X thus XA_4 is the diameter of (P') , we conclude that $XL \perp A_4L$.

Therefore L, T, X are collinear.

Moreover $PX// = P''A_4// = A_3T$ we obtain A_3XPT is a parallelogram. But A is the midpoint of PA_3 hence T, A, X are collinear.

This means $L \in AX$. Likewise, BY, CZ also pass through L. Our proof is completed.

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